mesh density since, as shown in Fig. 2, the velocity variation in the new composite coordinate is considerably reduced over that usually found in turbulent boundary layers.

Conclusion

In conclusion, a new composite coordinate transformation, based on well-known fluid dynamic concepts, has been presented which captures the boundary-layer thickness and simultaneously enlarges the wall-layer region. With this transformation, an adaptive grid procedure for numerical solution of the governing equations was demonstrated for laminar, transitional, and turbulent flow. The adaptive grid scheme presented here is simpler to use than a variable grid scheme since now only the total number of desired points needs to be specified by the user.

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Free Energy of Random Sound Oscillations in Gases

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Introduction

In solids and liquids, the effect of sound waves on the thermodynamic quantities was studied by Landau and others.[1] In ionized gases, the electrical oscillations (plasma oscillations) affect the thermal equilibrium of the system.[2,3] Similarly, we are considering the contribution of the sound-wave oscillations to the free energy of noncondensed gases. In thermal equilibrium of gases, the acoustic oscillations share the partition of energy and thus change the thermodynamic functions of the system. The distribution of the sound-wave quanta is determined by Bose statistics, which are used herein.

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The problem under consideration is concerned with gases as a macroscopic continuum which exhibits a set of separate elementary exitations, that is, the sound-wave oscillations. These exitations behave as "quasiparticles" moving in the volume occupied by the gas and have definite energies. The free energy of the gas evaluated by the theory to be presented will take into consideration the energy of the random sound-wave oscillations, in addition to the random thermal energies of the gas particles. It will be shown that the effect of the sound waves is important only at high temperatures and high gas densities. The results of this theory are applicable at temperatures and densities for which the gas is not in a condensed state (liquid or solid).

Although nonideal effects due to finite particle size are not taken into account explicitly, it should be noted that the gas under consideration is not an ideal one. The existence of sound waves in the gas implies that there are particle interactions, since a gas cannot perform the ordered, collective mean mass motions of random sound waves without such interactions.

Theory

Consider a gas as a continuum of volume V containing N atoms. Because the velocity of the gas in a sound wave is in the direction of propagation, the sound waves are longitudinal. Each oscillator of frequency ω_σ of the longitudinal sound waves can have only the energies

$$E_{n_\sigma}=\hbar\omega_\sigma(n_\sigma+\tfrac{1}{2}),\quad n_\sigma=0,1,2,\ldots\infty \tag{1}$$

where $\hbar=h/2\pi$ = reduced Planck constant.

The frequency of the sound waves with wave number k_σ is

$$\omega_\sigma=k_\sigma C_s,\quad C_s=(\gamma KT/M)^{1/2} \tag{2}$$

where $\gamma\cong C_p/C_v$ is the polytropic coefficient and M = mass of atoms. Accordingly, the partition function of the gas oscillations is

$$Z=\prod_\sigma\sum_{n_\sigma=0}^{\infty}e^{-\beta\hbar C_s k_\sigma(n_\sigma+1/2)}=\prod_\sigma\frac{e^{-\beta\hbar C_s k_\sigma/2}}{1-e^{-\beta\hbar C_s k_\sigma}} \tag{3}$$

in which $\beta=1/KT$, where K is the Boltzmann constant and T the temperature of the system. From the partition function Z, the thermodynamic quantities such as pressure, internal energy, etc. are derived in the usual way. The free energy of the random sound oscillations is given by

$$\Delta F=-KT\ln Z \tag{4}$$

In the limit $V\to\infty$, the discrete eigenfrequencies ω_σ are replaced by a continuous spectrum $\omega=\omega(k)$ in accordance with the dispersion law for sound waves of wave length $\lambda=2\pi/k$,

$$\omega=kC_s,\qquad 0\le k\le\hat{k} \tag{5}$$

The theory to be presented is sensitive toward the cutoff wave number $\hat{k}$, which is large in all cases of interest. Since acoustic waves with wavelengths $\lambda<\max(\bar{r},L)$ and mean free paths $L<\bar{r}=n^{-1/3}$ are not possible in gases, $\hat{k}=2\pi/\hat{\lambda}$ is determined by the mean free path L,

$$\hat{k}=2\pi/L,\qquad L=L(n,T) \tag{6}$$

where $n=N/V$ is the density of the atoms. The number of (longitudinal) wave modes with wave numbers between k and $k+dk$ in volume V is $g(k)dk=V4\pi k^2dk/(2\pi)^3$. Accordingly, Eqs. (3) and (4) give

$$\Delta F=-\frac{KTV}{2\pi^2}\int_0^{\hat{k}}k^2\ln\left(\frac{e^{-\beta\hbar C_s k/2}}{1-e^{-\beta\hbar C_s k}}\right)dk \tag{7}$$

The integral in Eq. (7) is decomposed into the contributions from the ground state ($n_\sigma=0$) and the higher states ($n_\sigma>0$). By Eqs. (3) and (7)

$$\Delta F=F_1+F_2 \tag{8}$$

where

$$F_1=\frac{V\hbar C_s}{4\pi^2}\int_0^{\hat{k}} k^3 \mathrm{d}k = V\hbar C_s\left(\frac{\hat{k}^2}{4\pi}\right)^2 \tag{9}$$

and

$$F_2=\frac{V\hat{k}^3}{6\beta\pi^2}\ln(1-e^{-\beta\hbar C_s\hat{k}})-\frac{V\hbar C_s}{6\pi^2}\int_0^{\hat{k}}\frac{k^3\mathrm{d}k}{e^{\beta\hbar C_s k}-1} \tag{10}$$

by partial integration. The integral in Eq. (10) can be solved for "high" and "low" temperatures by series expansions which give[4,5]

$$F_2=\frac{Vk^3}{6\beta\pi^2}\left\{\ln(1-e^{-x})-\frac{1}{3}+\frac{x}{8}-\sum_{\nu=1}^{\infty}\frac{B_{2\nu}x^{2\nu}}{(2\nu+3)2\nu!}\right\},\quad x<2\pi \tag{11}$$

and

$$F_2=\frac{V\hat{k}^3}{6\beta\pi^2}\left\{\ln(1-e^{-x})-\frac{1}{x^3}\left[6\zeta(4) -x^4\sum_{n=1}^{\infty}e^{-nx}\left(\frac{1}{nx}+\frac{3}{n^2x^2}+\frac{6}{n^3x^3}+\frac{6}{n^4x^4}\right)\right]\right\},\quad x\geq 1 \tag{12}$$

where

$$x=\beta\hbar C_s\hat{k} \tag{13}$$

and $B_{2\nu}$ are the Bernoulli numbers and $\zeta(4)=\pi^4/90$ is the Riemann ζ function. For comparison purposes, ΔF and the classical free energy F_0 of the ideal monatonic gas (M = atomic mass) are stated as

$$F_0=NKT\left\{\ln\left[\left(\frac{\hbar^2}{2\pi MKT}\right)^{3/2}n\right]-1\right\} \tag{14}$$

and

$$\Delta F=V\hbar C_s\left(\frac{\hat{k}^2}{4\pi}\right)^2+\frac{V\hat{k}^3}{6\beta\pi^2}\left[\ln(1-e^{-x})-\frac{1}{3}+\frac{x}{8}-\sum_{\nu=1}^{\infty}\frac{B_{2\nu}x^{2\nu}}{(2\nu+3)2\nu!}\right],\quad x<2\pi \tag{15}$$

$$\Delta F=V\hbar C_s\left(\frac{\hat{k}^2}{4\pi}\right)^2+\frac{V\hat{k}^3}{6\beta\pi^2}\left\{\ln(1-e^{-x})-\frac{1}{x^3}\left[6\zeta(4) -x^4\sum_{n=1}^{\infty}e^{-nx}\left(\frac{1}{nx}+\frac{3}{n^2x^2}+\frac{6}{n^3x^3}+\frac{6}{n^4x^4}\right)\right]\right\}\quad x\geq 1 \tag{16}$$

for "high" and "low" temperature, respectively.

It is interesting to compare the specific heat of the ideal gas C_0 and the sound oscillations ΔC in the high-temperature limit, $x=\beta\hbar C_s\hat{k}\ll 1$. By Eqs. (14) and (15),

$$C_0=-\frac{T\partial^2F_0}{\partial T^2}=\frac{3NK}{2} \tag{17}$$

$$\Delta C=-\frac{T\partial^2\Delta F}{\partial T^2}\sim\frac{2\pi}{3}(V/L^3)K\ll C_0 \text{ for } \bar{r}=n^{-1/3}\ll L \tag{18}$$

In the derivation of Eq. (18), it should be noted that $\Delta F\cong(V\hat{k}^3/6\beta\pi^2)\ln x$ for $x\ll 1$ where $x\propto T^{-1/2}$. It is seen that $\Delta C<C_0$ at high temperatures since $\bar{r}<L$.

This completes the mathematical aspects of the problem, the physical implications of which will be discussed next.

Discussion

In a hypothetical ideal equilibrium gas without particle interactions, the average particle velocity is $\langle c\rangle=\int cf(c)\mathrm{d}^3c=\mathbf{0}$, i.e., the particles have pure thermal velocities c with a Maxwell distribution $f(c)$. No random mean mass motions or collective particle motions, such as sound oscillations, exist due to the absence of particle interactions. The free energy of the ideal or noninteracting monatomic gas is therefore F_0 [Eq. (14)]. In any real gas with particle interactions, stochastic mean mass motions $\langle v(r,t)\rangle=\int vf(v,r,t)\mathrm{d}^3v\neq\mathbf{0}$ exist due to the presence of thermally excited sound waves [$f(v,r,t)$ is the local distribution of actual particle velocities $v=\langle v\rangle+c$]. Since the total energy of the gas is distributed both over the thermal particle motion c and the stochastic, acoustic mean mass motions $\langle v(r,t)\rangle$, a free-energy contribution from the random sound waves exists. Thus, the free-energy ΔF of the random sound oscillations represents a nonideal effect which is ultimately due to particle interactions, which make a hydrodynamic or continuum description of a gas possible.

In the contribution ΔF of the sound waves to the free-energy F of an ideal gas as given by Eq. (15) or (16), we identify two parts: 1) F_1, the contribution of the "zero oscillation" mode which corresponds to $n_\sigma=0$ in Eq. (1); and 2) F_2, the higher mode oscillation contributions $n_\sigma>1$. The explanation for the increase of the free energy of the sound quanta $\hbar\omega$ with temperature is given by statistics. In the high-temperature limit, the number N_ω of sound quanta of frequency ω is $N_\omega\cong KT/\hbar\omega$, i.e., increases proportional with T.

In Fig. 1 we have drawn $\Delta F/F_0$ for monatomic helium gases over a range of temperatures and densities to show the variation of ΔF. The overall contribution of ΔF is larger at higher densities but decreases rapidly for lower densities, especially at high temperatures. Quantitatively, ΔF represents a noticeable effect only at extremely high densities n of the gas ($n>10^{21}\mathrm{cm}^3$). For this reason, the free-energy ΔF of the

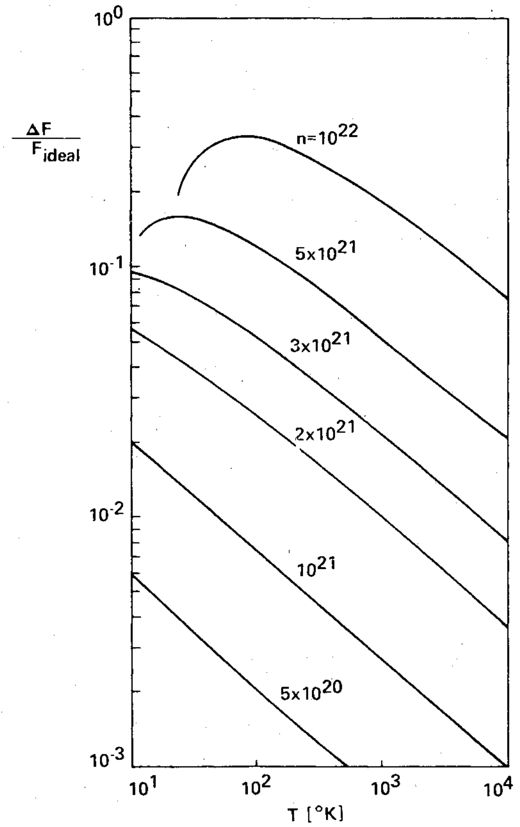

Fig. 1 Free energy ΔF due to sound waves in helium gas as function of temperature T(K) and density n(cm^{-3}).

sound oscillations has to be considered in the evaluation of the thermodynamic functions of high-temperature gases only at high densities.

Gases with a considerable acoustic noise background are encountered in various high-temperature engineering systems, such as gas turbines, jet engines, rocket exhausts, etc. The theory presented permits calculation of the free energy ΔF of the acoustic degrees of freedom in such systems, provided that the acoustic noise is in thermal equilibrium. Considerably larger free-energy contributions are to be expected under nonequilibrium conditions, particularly if the acoustic fluctuations exhibit intensity levels corresponding to turbulence.

Acknowledgment

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Higher Order Sensitivities in Structural Systems

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Introduction

ALTHOUGH several papers on sensitivity analysis have been published in the last few years, there has been little interest in higher order sensitivities. Higher order sensitivities are especially important if the relation between finite difference and differential sensitivities or an expression for the direct calculation of difference sensitivities is not available. This is, as far as we know, the case for difference sensitivities of natural frequencies and mode shapes. In this Note a flexibility method for the calculation of higher order sensitivities of flexibilities, structural eigenvalues, and flexibility modes is introduced. If the first-order and a number of higher order sensitivities are available, an approximation of the difference sensitivities based on a Taylor series becomes possible.

Most of the papers dealing with sensitivity analysis concern only first-order differential sensitivities of eigenvalues and eigenvectors. These derivatives are used to approximate the finite difference or large-change sensitivities. The authors use the stiffness matrix formulation and assume that the stiffness matrices are assemblages of matrices of the component elements.[1-5]

Since finite difference sensitivities are in general nonlinear functions of the structural parameters, the derivatives cannot yield good approximations if large parameter changes are considered. With the help of truncated Taylor series and a number of higher order sensitivities a better approximation is possible. An exact and simple matrix equation for the calculation of the large-change sensitivity of eigenvalues and eigenvectors is still not available.

The widely used stiffness methods are of limited utility in the important field of experimental modal analysis. In this case data for partial flexibility matrices are measured by Fourier analyzers. Curve-fitting techniques are used to extract the eigenvalues and flexibility modes (or residues). Only a limited number of important modes is considered.

The assumption that the static stiffness matrix, which appears in the system equation, consists of a superposition of several element matrices is not always valid, even in computer-aided design. If the mass is concentrated in the nodes and mass moments of inertia are not taken into account, the full stiffness matrix must be reduced to a "pseudostiffness" matrix. When the rotational degrees of freedom are eliminated, the linear characteristic of the stiffness matrix with respect to the element stiffnesses is lost.

In both cases (modal analysis and computer-aided design with a simple lumped-mass model) the classical methods are not very useful. Some methods developed in system and electrical network theory can be generalized and transferred successfully to mechanical structural sensitivity analysis.[6]

Flexibility Sensitivities

Consider the structure represented in Fig. 1. This linear structure consists of two connected parts: a nonvariable substructure and a variable element. Suppose that the variable element is characterized by a stiffness matrix $|K_{ll}|$ which is a function of at least one parameter R_m. For the entire structure the flexibility matrix $|S'|$ with partial matrices $|S'_{rq}|$ will be used as a model.

The difference sensitivity of a partial flexibility matrix $|S'_{rq}|$ for a (large) element parameter change ΔR_m can be written as

$$\frac{\Delta |S'_{rq}|}{\Delta R_m} = -|S'_{rl}|\frac{\Delta |K_{ll}|}{\Delta R_m}[|I| + |S'_{ll}|\Delta |K_{ll}|]^{-1}|S'_{lq}| \qquad (1)$$

The evidence is given in Ref. 6. Differential sensitivities become equal to difference sensitivities if infinitesimal changes are introduced. A limit transformation results immediately in

$$\frac{\partial |S'_{rq}|}{\partial R_m} = -|S'_{rl}|\frac{\partial |K_{ll}|}{\partial R_m}|S'_{lq}| \qquad (2)$$

In the special case that

$$\frac{\Delta |K_{ll}|}{\Delta R_m} = \frac{\partial |K_{ll}|}{\partial R_m} \qquad (3)$$

the relation between difference and differential sensitivities for some partial matrices of the flexibility matrix can be

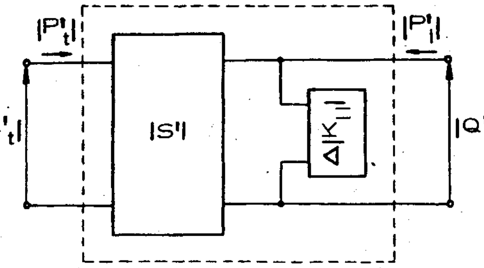

Fig. 1 Generalized *n* port representation of a modified structure.

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